TAUP 2025 XICHANG SICHUAN, CHINA 2025.8.24 - 8.30

Imaginarity in Neutrino Systems: A Resource-Theoretic Perspective

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Based on arXiv:2412.01871

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Complex Numbers in Quantum Theory

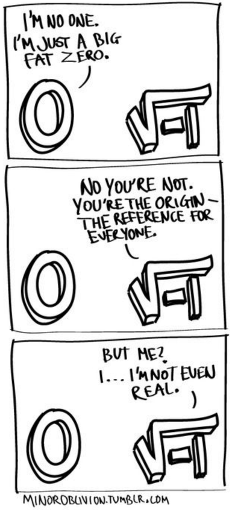
In quantum mechanics, wave-particle duality makes complex numbers essential for describing a system's **state, dynamics, and interactions**.

The postulates of quantum mechanics state that a quantum system is described by a **complex Hilbert space**.

Key Questions

- Can quantum physics be reformulated using only *real* numbers?
- Are complex numbers used merely for mathematical *convenience* or are they essential?

It is therefore essential to thoroughly investigate the role of complex numbers in quantum systems!



Motivation

1. Given the unique significance of imaginary numbers in quantum theory, recent developments have led to a comprehensive formulation of imaginarity in **quantum physics** and **quantum information theory**

"Quantifying the Imaginarity of Quantum Mechanics" – J. Phys. A 2018
"Operational Resource Theory of Imaginarity" – PRL 2021

2. **Resource theory of imaginarity** quantifies and utilizes imaginary components for information processing
3. Related to **resource theory of coherence**, as it is basis dependent
4. Closely linked to **quantum coherence** since imaginary parts appear in off-diagonal density matrix elements

Neutrinos, existing as natural superpositions of mass eigenstates, provide an ideal physical system to test **imaginarity** through *oscillation dynamics*!

Measures of Imaginarity

- An imaginarity measure quantifies the **amount of imaginary components** in a quantum state
- For an imaginarity measure, denoted as $\mathcal{I}$, to serve as a meaningful and consistent tool, it **must** satisfy several criteria:

Faithfulness: The measure of imaginarity should be non-negative for any quantum state, i.e., $\mathcal{I}(\rho) \geq 0$.

Imaginarity Monotonicity: The measure of imaginarity should not increase under the application of real operations, i.e., if ϕ is a real channel, then $\mathcal{I}[\phi(\rho)] \leq \mathcal{I}(\rho)$

1 ℓ_1-norm of imaginarity

Encapsulates the essence of imaginarity by summing the absolute values of the imaginary components of the off-diagonal elements in the density matrix.

$$\mathcal{I}_{\ell_1}(\rho) = \sum_{i \neq j} |\mathrm{Im}(\rho_{ij})|$$

2 Relative entropy of imaginarity

Measures the entropic distance between the quantum state and its real counterpart.

$$\mathcal{I}_r(\rho) = S(\mathrm{Re}(\rho)) - S(\rho)$$

where $S(\rho) = -\mathrm{Tr}[\rho \log \rho]$ is defined as the von Neumann entropy.

Neutrinos as Quantum Superpositions

- The *mixing* between the flavor and mass eigenstates is described by:

$$\left|\nu_\alpha^h(l)\right\rangle = \sum_{i=1}^{3} U_{\alpha i}^* \left|\nu_i^h(l)\right\rangle$$

- The observation of **flavour oscillations** implies that neutrinos are massive.
- This may result in the generation of a small but finite magnetic dipole moment through higher-order **quantum loop corrections**.

- Since the *helicity* mass eigenstates are not stationary in the presence of an external magnetic field, they must be decomposed into the stationary *spin* mass eigenstates:

$$\left|\nu_i^h(l)\right\rangle = \sum_{s=-1}^{+1} k_i^s \left|\nu_i^s(l)\right\rangle$$

- For the three-flavour mixing, the evolution of the flavour eigenstate $\left|\nu_\alpha^L(l)\right\rangle$ undergoing SFOs can be expressed as:

$$\left|\nu_\alpha^L(l)\right\rangle = \sum_{h'=L}^{R} \sum_{\beta=e}^{\tau} \sum_{i=1}^{3} \sum_{s=-1}^{+1} U_{\alpha i}^* U_{\beta i} C_{is}^{Lh'} e^{-iE_i^s t} \left|\nu_\beta^{h'}\right\rangle$$

where $C_{is}^{Lh'} = \langle \nu_i^{h'} | \hat{P}_s^i | \nu_i^L \rangle$ is the plane-wave expansion coefficient, and $\hat{P}_s^i = \left|\nu_i^s\right\rangle \left\langle \nu_i^s\right|$ is the projection operator.

Evolution of Two-flavour Neutrino Systems

The evolution of the flavor eigenstate $\left|\nu^L_{e,\mu}\right\rangle$, in spin-flavor basis, can be written as:

$$\left|\nu^L_{e,\mu}(\theta,l)\right\rangle = f_1(\theta,l)\left|\nu^L_{e,\mu}\right\rangle + f_2(\theta,l)\left|\nu^R_{e,\mu}\right\rangle + f_3(\theta,l)\left|\nu^L_{\mu,e}\right\rangle + f_4(\theta,l)\left|\nu^R_{\mu,e}\right\rangle$$

where

$$\begin{aligned}
2f_1(\theta,l) &= \cos^2\theta\, e^{-iE_i^+ l} + \sin^2\theta\, e^{-iE_j^+ l} + \cos^2\theta\, e^{-iE_i^- l} + \sin^2\theta\, e^{-iE_j^- l}\\
2f_2(\theta,l) &= \cos^2\theta\, e^{-iE_i^- l} + \sin^2\theta\, e^{-iE_j^- l} - \cos^2\theta\, e^{-iE_i^+ l} - \sin^2\theta\, e^{-iE_j^+ l}\\
2f_3(\theta,l) &= -\sin\theta\cos\theta\left[e^{-iE_i^+ l} - e^{-iE_j^+ l} + e^{-iE_i^- l} - e^{-iE_j^- l}\right]\\
2f_4(\theta,l) &= -\sin\theta\cos\theta\left[e^{-iE_i^- l} - e^{-iE_j^- l} - e^{-iE_i^+ l} + e^{-iE_j^+ l}\right]
\end{aligned}$$

The evolution of standard two-flavor FOs can be derived by setting the spin eigenvalues equal and eliminating the spin-flipping dof!

From the state evolution, we can deduce the **4 × 4** density matrix $\rho^{SFO}_{e,\mu}(\theta,l) = \left|\nu^L_{e,\mu}(\theta,l)\right\rangle\left\langle\nu^L_{e,\mu}(\theta,l)\right|$ and **2 × 2** density matrix $\rho^{FO}_{e,\mu}(\theta,l) = |\nu_{e,\mu}(\theta,l)\rangle\langle\nu_{e,\mu}(\theta,l)|$.

Imaginarity in Flavour Oscillations (FO)

- The ℓ_1-norm of imaginarity can be calculated using the density matrices:

$$\mathcal{I}_{\ell_1}\left(\rho^{\text{FO}}_{e,\mu}\right) = \sum_{i\neq j}\left|\text{Im}\left(\rho^{\text{FO}}_{e,\mu}\right)_{ij}\right| = \left|\sin 2\theta \, \sin\left(\frac{\Delta m^2_{ji}\, l}{2E}\right)\right|$$

- Despite the initial neutrino states being different, the ℓ_1-norm of imaginarity turns out to be the **same** for them
- ℓ_1-norm of imaginarity is **nonzero** even for two-flavor FOs
- This underscores the significance of imaginarity as a resource coming from the **intrinsic propagation dynamics** of neutrinos
- To calculate the relative entropy of imaginarity $\mathcal{I}_r(\rho)$ in two-flavor FOs, we evaluate:

$$|\langle \nu^*_{e,\mu}(\theta, l)|\nu_{e,\mu}(\theta, l)\rangle| = \sqrt{\cos^4\theta + \sin^4\theta + 2\sin^2\theta\cos^2\theta\cos\left(\frac{\Delta m^2_{ji}\, l}{E}\right)}$$

Both observables $\mathcal{I}_{\ell_1}(\rho)$ and $\mathcal{I}_r(\rho)$ will be nonzero only if the quantum state ρ contains imaginary components, thereby serving as effective metrics of the imaginarity in the quantum state!

Neutrino Spin-Flavour Oscillations (SFOs)

- Additional spin-flip degrees of freedom emerge from the neutrino **magnetic moment** which can interact with magnetic fields and undergo SFOs [Giunti & Studenikin, Rev.Mod.Phys. 2015]
- In the minimally extended Standard Model (with right-handed neutrinos), the diagonal magnetic moments of massive Dirac neutrinos can be calculated to be:

$$\mu_\nu \simeq 3.2 \times 10^{-19} \left(\frac{m_i}{1\,\mathrm{eV}} \right) \mu_B$$

- In the *ultra-relativistic limit*, with equal magnetic moments for all neutrino states, the **spin-flavour oscillation phase** is given by:

$$\xi_{ji}^{s's} = E_j^{s'} - E_i^{s} = \frac{\Delta m_{ji}^2}{2E} + \mu_\nu (s' - s) B_\perp$$

- Quantum coherence in SFOs persists over **astrophysical distances!** [Alok *et al*, PRD 2025]

Similar to the ℓ_1-norm of coherence, the ℓ_1-norm of imaginarity is also a basis-dependent measure -*must be evaluated for SFOs*!

Imaginarity in SFOs

- The ℓ_1-norm of imaginarity can be calculated as:

$$\mathcal{I}_{\ell_1}(\rho^{\text{SFO}}_{e,\mu}) = |\sin\theta\cos\theta| \left\{ \left| 2(\cos^2\theta - \sin^2\theta)\sin(\xi^{+-}_{ii} l)\cos\left(\frac{\Delta m^2_{21}}{2E} l\right)\right| + \left|\sin\left(\frac{\Delta m^2_{21}}{2E} l\right)\right| \left(\left|\cos(\xi^{+-}_{ii} l) - 1\right| + \left|\cos(\xi^{+-}_{ii} l) + 1\right| \right) \right\}$$

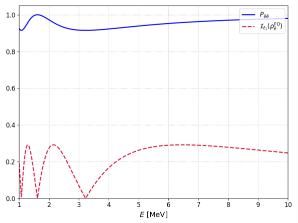
- The relative entropy of imaginarity can be calculated from:

$$\left|\langle \nu^{L*}_{e,\mu}(\theta, l) | \nu^{L}_{e,\mu}(\theta, l)\rangle\right| = \left[\frac{\cos^4\theta + \sin^4\theta}{2}(1 + \cos(2\xi^{+-}_{ii} l)) + \frac{\sin^2\theta\cos^2\theta}{2} \left\{ \cos(2\xi^{+-}_{ji} l) + \cos(2\xi^{-+}_{ji} l) + 2\cos\left(\frac{\Delta m^2_{ij}}{E} l\right)\right\}\right]^{\frac{1}{2}}$$

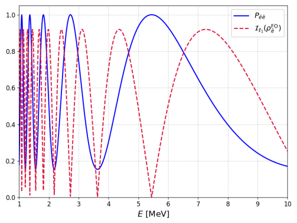
In the absence of spin-flipping induced by the interaction of neutrinos with an external magnetic field, both measures for the SFO system reduce to those of the FO system!

Imaginarity Dynamics in Reactor Neutrino Experiments

- **Nuclear reactors** provide intense sources of **coherent antineutrino fluxes**, enabling precision studies of oscillation parameters.
- Daya Bay $\rightarrow$ short-baseline (~ 1.5 km), $E = 1$–10 MeV.
- KamLAND $\rightarrow$ medium-baseline (~ 180 km), similar E range.
- We focus on these experiments since the analysis considers neutrino oscillations in vacuum.



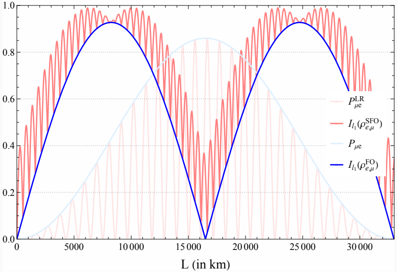
Daya Bay

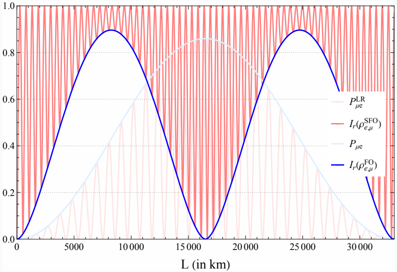


KamLAND

- $\mathcal{I}_{\ell_1}^{\rm FO}$ peaks where the survival probability P_{ee} changes **most rapidly**, indicating **maximal quantum interference**.
- It **vanishes** when $P_{\bar{e}\bar{e}}$ reaches its **extrema** (maxima or minima), where the neutrino state realigns with a flavor eigenstate and becomes **stationary** with no mixing.

Imaginarity Dynamics in GeV Muon Neutrinos



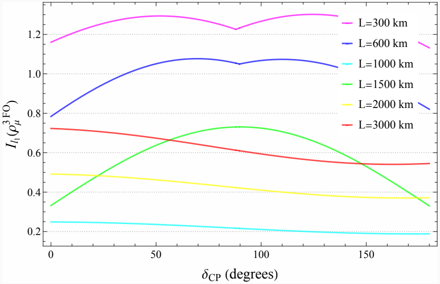


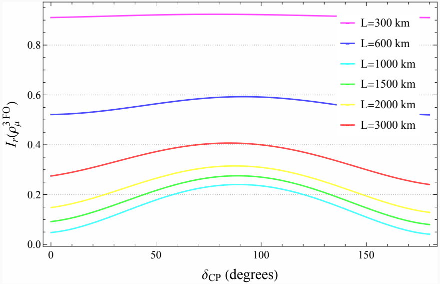
- The oscillatory behaviour shows that imaginarity varies continuously with **propagation distance**.
- For SFO, the extrema of the imaginarity measures are found to align closely with those of FOs.
- The imaginary component of the density matrix reaches its maximum when the transition and survival probabilities are approximately equal, averaging around **half!**
- Imaginarity **vanishes** at the turning points of the FO probability curve, where the neutrino mass eigenstate **aligns** with a flavor eigenstate.
- This indicates that the maximum amount of imaginarity enters the neutrino system when the oscillations are **least deterministic**.
- Exhibits a **similar** trend to that observed in reactor neutrino experiments.

Imaginarity Dynamics in Three-Flavour Systems

- The complex phase δ_{CP} appears in the PMNS mixing matrix and is anticipated to contribute to the imaginarity measures.
- Similar to the case of two-flavour FOs, the imaginarity measures remain **non-zero**, even when $\delta_{CP} = 0$.
- Both the ℓ_1-norm and the relative entropy of imaginarity can be **enhanced or suppressed** compared to the case of $\delta_{CP} = 0$.
- For certain values of L, the dependence of imaginarity on δ_{CP} is minor, while for others, the deviation from that of $\delta_{CP} = 0$ could be **significant**.

This reaffirms that imaginarity is embedded in the dynamics of the neutrino system and persists even if the value of the CP-violating phase in the leptonic sector is zero!





Key Findings

Imaginarity in Two-Flavour Systems

For the first time, we have quantified imaginarity in neutrino systems using the ℓ_1-norm and relative entropy of imaginarity.

We have demonstrated that imaginarity persists as nonzero even in the context of two-flavor mixing, applicable to both flavor oscillations and spin-flavor oscillations.

Beyond CP Phase

Imaginarity as a resource in neutrino systems is not exclusively dependent on the CP phase.

The imaginarity as a resource can arise in the neutrino system, from the intrinsic quantum dynamics of the neutrino mixing itself.

Quantum Resource

Our work establishes imaginarity as a quantifiable resource in neutrino physics, opening new avenues for understanding quantum aspects of these fundamental particles.

How to measure it remains an open question!

Thank you for your attention!